Complexity, Entanglement, and Stabilizer Entropy

Alioscia Hamma

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TII - Abu Dhabi

Università degli Studi di Napoli Federico II

This is our 800th anniversary!

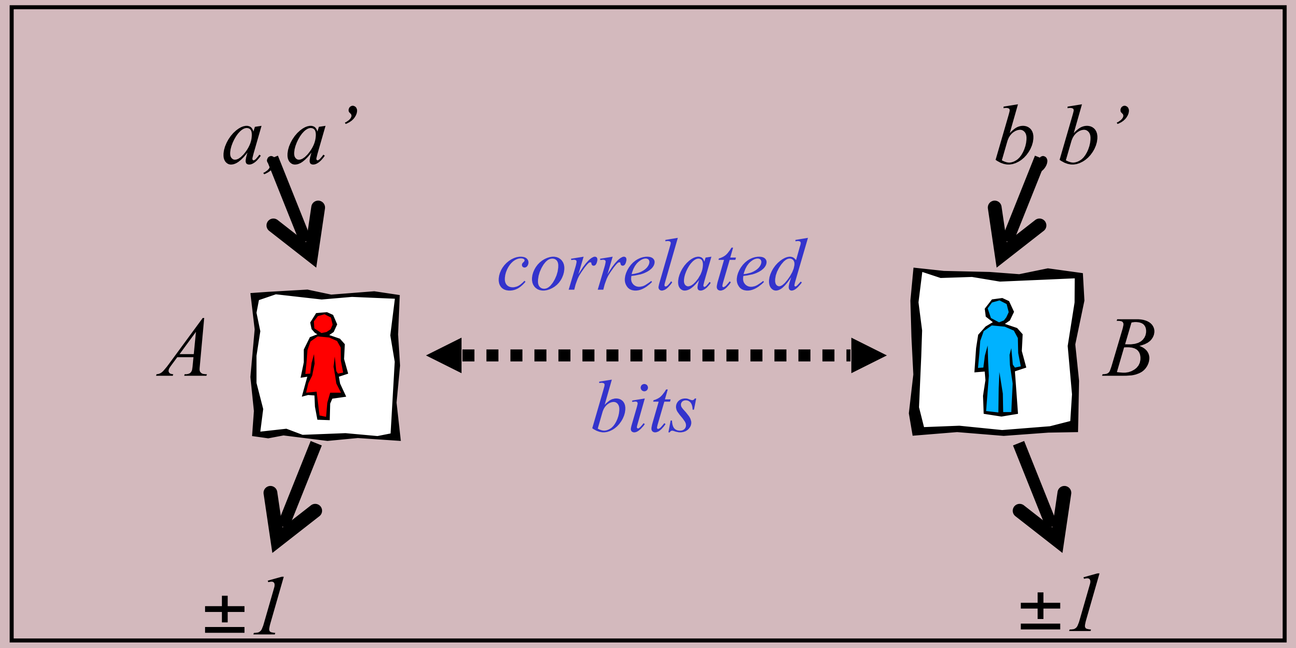
Quantumness, Entanglement and non-Stabilizerness

What makes a system quantum?

Quantumness

- Probabilistic theory (shared with classical)
- Incompatible observables (uncertainty principle)
- entangled states
- Bell's inequalities
- quantum advantage and universal quantum computers
- quantum chaos
- special properties at equilibrium and not of quantum many-body systems - quantum simulation

Bell's inequalities (CHSH)



$$|\langle ab\rangle + \langle a'b\rangle + \langle ab'\rangle - \langle a'b'\rangle| \leq 2$$

We know that there are *separable* measurements that violate the CHSH inequality

Why? Entanglement (?)

Entanglement

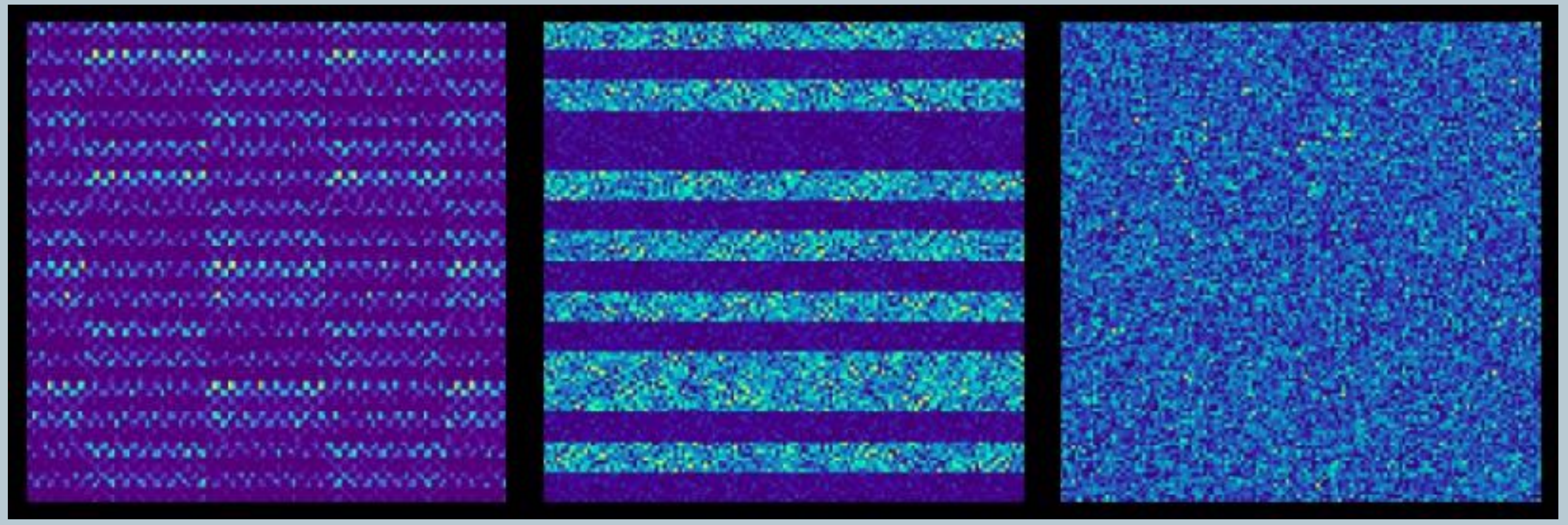
$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \qquad \psi_A = \mathrm{Tr}_B \psi$$

Not all density matrices are separable

Rényi entropies

$$S(\psi_A) = \frac{1}{1-\alpha} \log \mathrm{Tr} \psi_A^\alpha$$

Is entanglement enough?

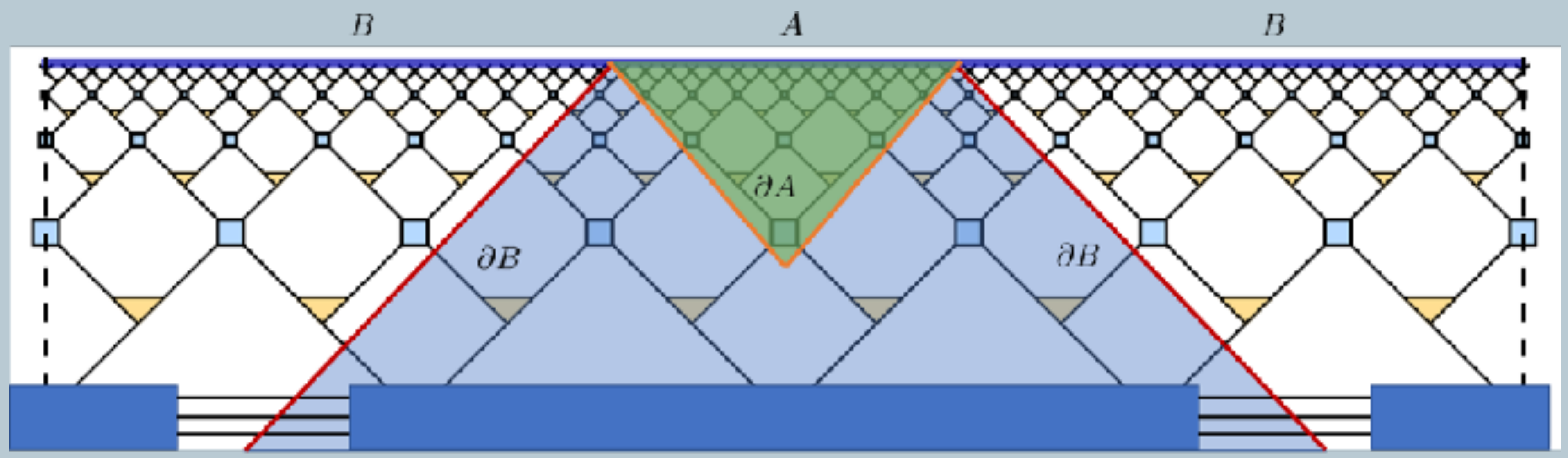


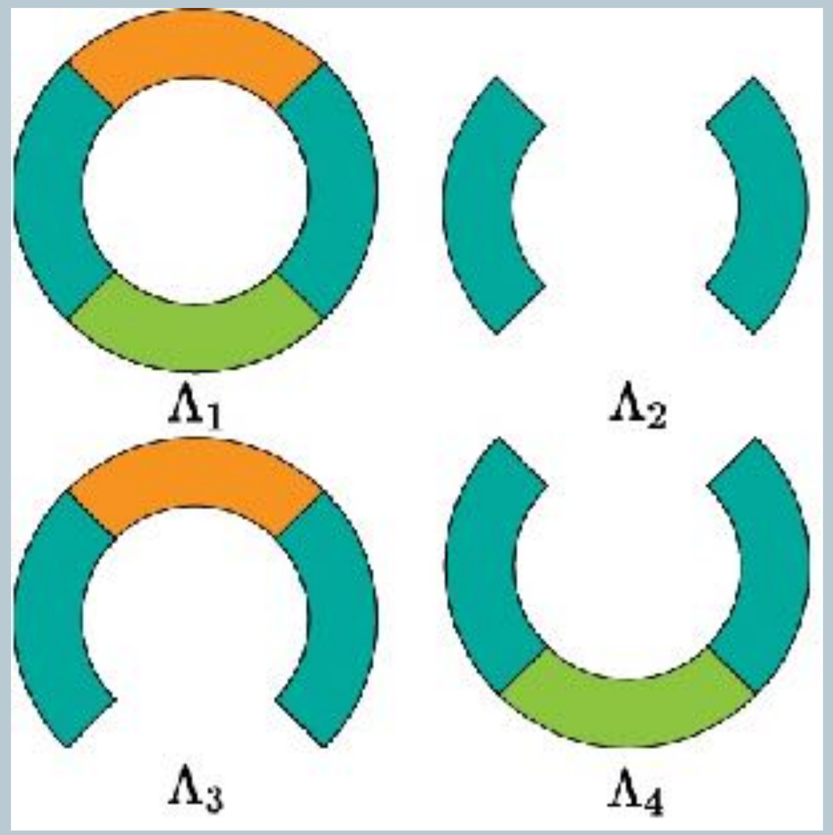
All these evolutions maximally entangle the state, but they feature increasing complexity

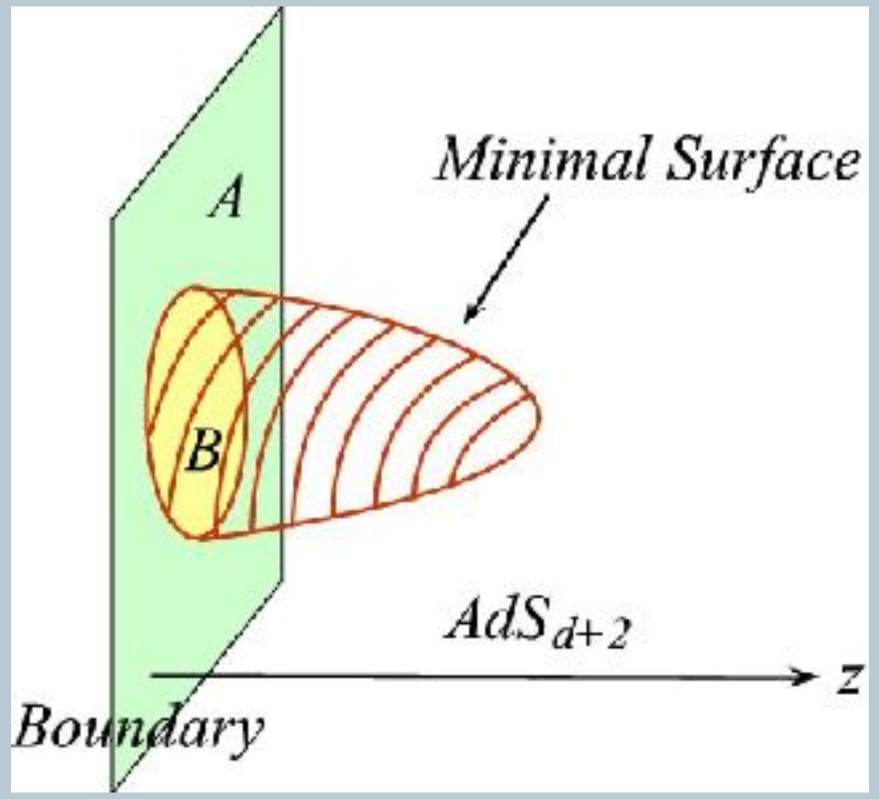
Entanglement in quantum many-body systems...and?

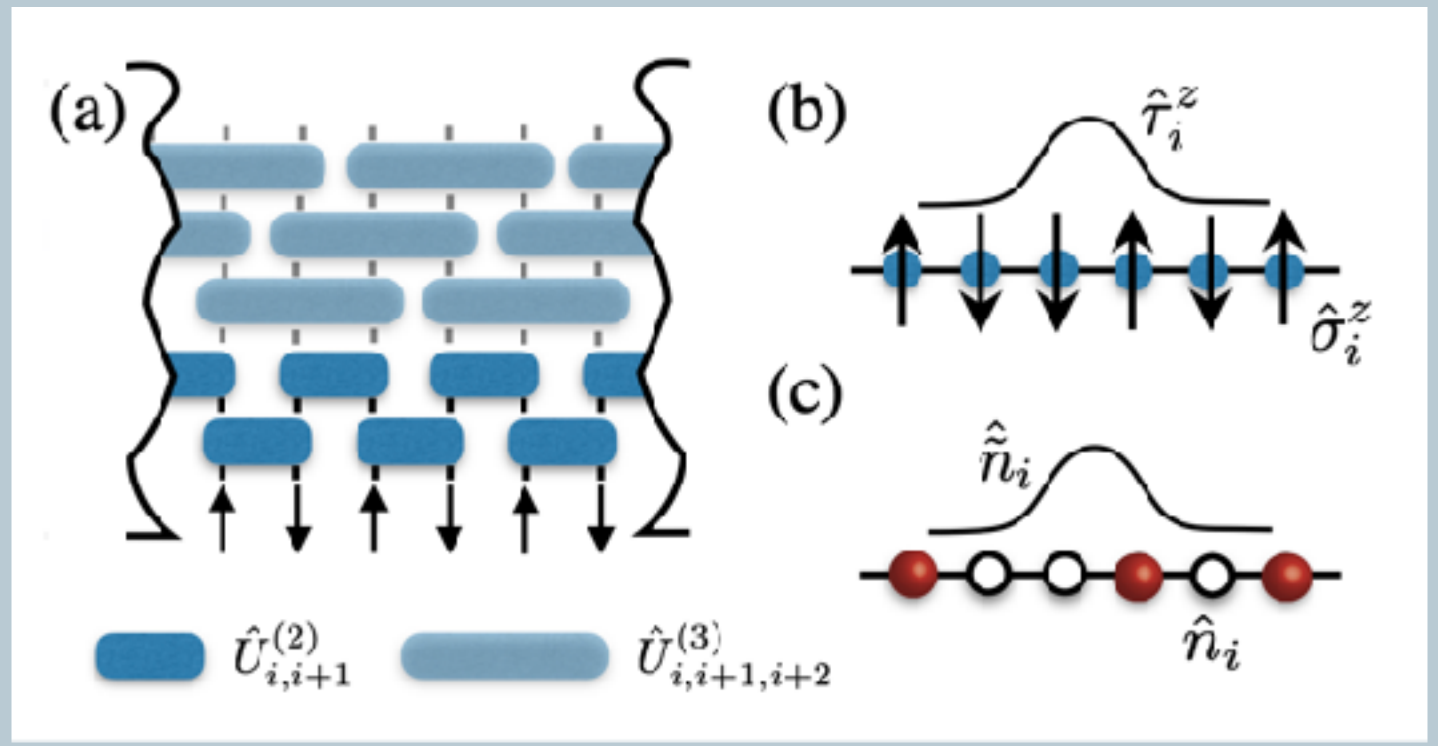
Entanglement is of fundamental importance in QMB systems because:

1. It determines the hardness of simulation by tensor network methods
2. It detects exotic phases of matter like topological states
3. explains the structure of CFTs
4. gives an explanation of thermalization in closed quantum systems



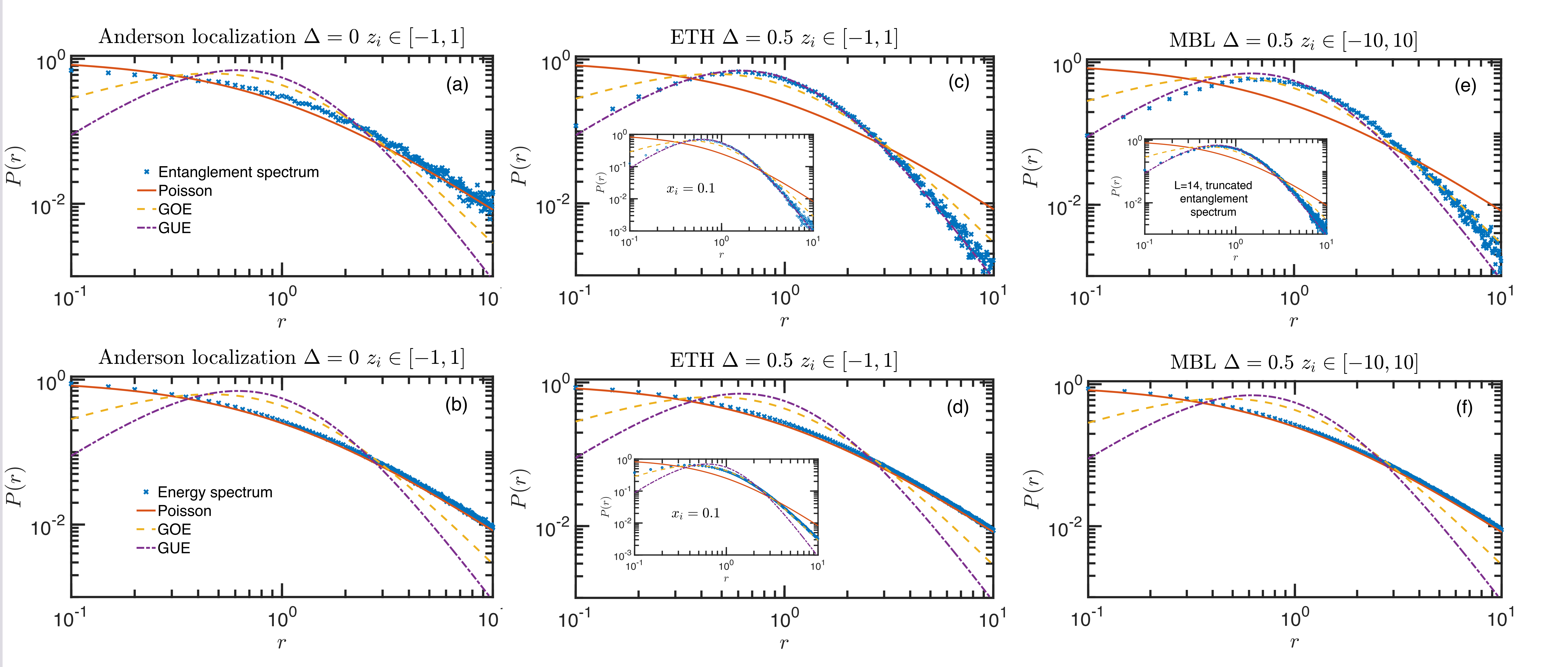






How complex is this entanglement?

Entanglement Complexity



Entanglement Spectrum Statistics shows different patterns of complexity of entanglement described by RMT and distinguishing ETH, Integrable models and MBL

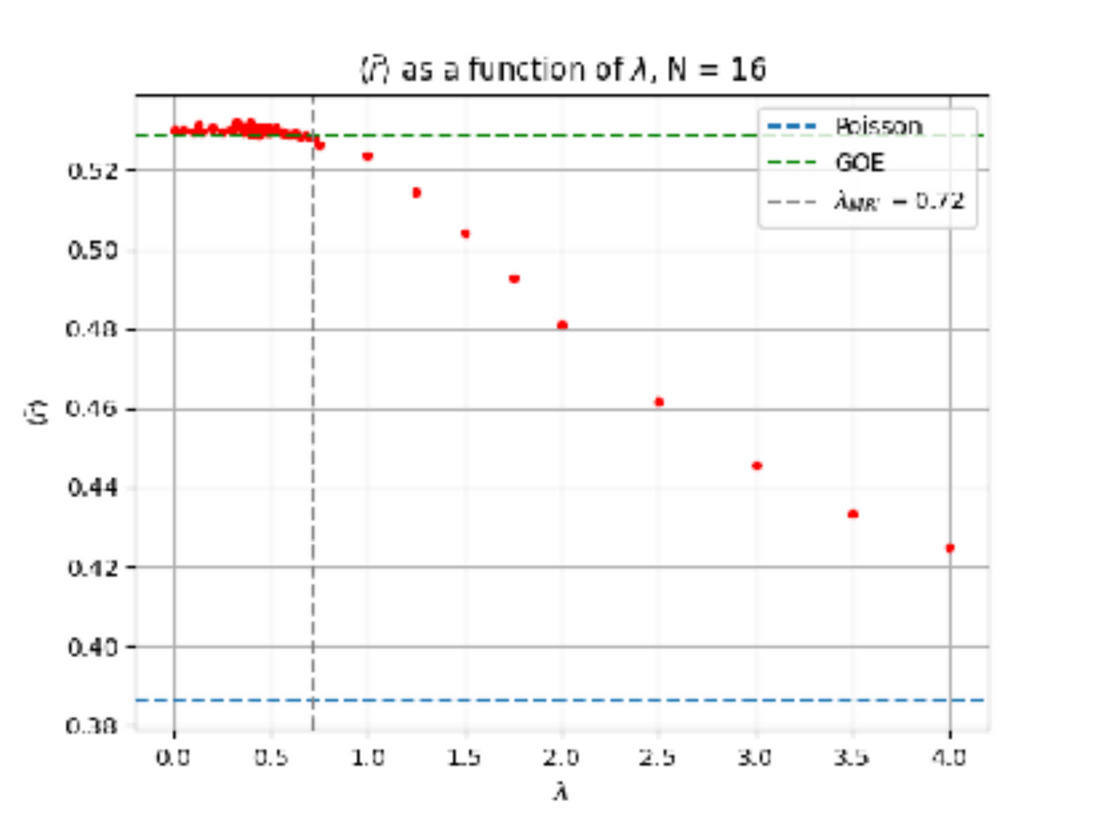
$$\delta_j = \lambda_{j+1} - \lambda_j$$

$$r_k = \frac{\delta_{k+1}}{\delta_k}$$

$$\tilde{r}_k = \frac{\min[\delta_{k+1}, \delta_k]}{\max[\delta_{k+1}, \delta_k]}$$

Spectral rigidity in RK-sign model

$$|\psi(\lambda)\rangle = \frac{1}{\sqrt{Z(\lambda)}} \sum_\sigma W_\sigma e^{-\lambda E_\sigma} |\sigma\rangle$$



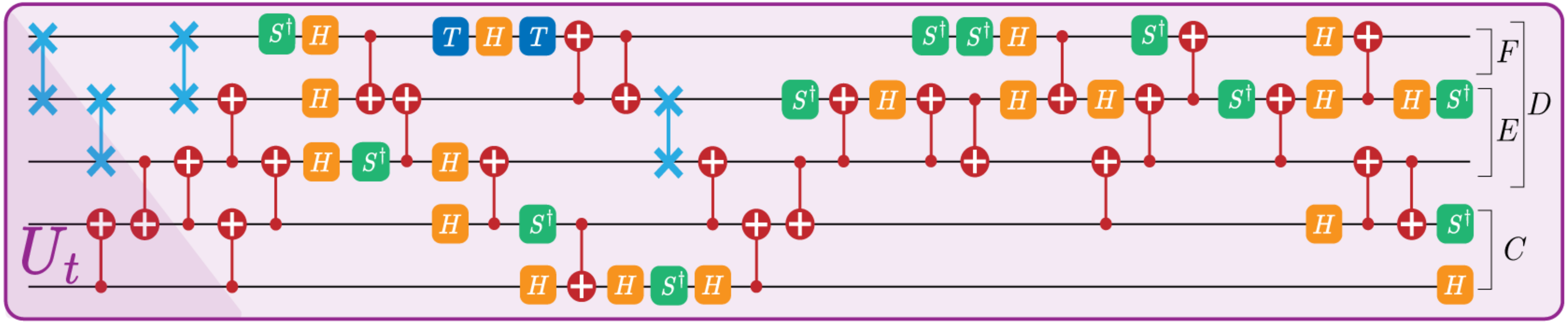
Phys. Rev. Lett. 112, 240501 (2014);

Phys. Rev. B 96, 020408 (2017)

Phys. Rev. B 107, 134202 (2023)

Quantum circuits can be very complex and thus hard to simulate

both learning and simulating are exp(2n) costly. Of course, it would be easy without entangling gates



FACT: circuits with only H, S, C_X are benign: why?

Quantum circuits *can* be very simple

These stringy objects are easy to manipulate, we need 4n bits

$$X_1 I_2 Z_3 Z_4 Y_5 X_6 \ldots Y_n$$

These processes are easy, n^2 matrix elements

$$HXH = Z \qquad HZH = X$$

$$CX_1C = X_1X_2 \qquad ZXZ = -X$$

$$SXS^\dagger = Y \qquad CZ_2C = Z_1Z_2$$

Clifford group: centralizer of Pauli group. Strings are mapped into strings

$$P \mapsto C^\dagger P C = P'$$

Gottesman et al : Clifford group is easy to both learn and simulate: poly(n)

Simulating t-doped circuits ~ poly (n) exp (t)

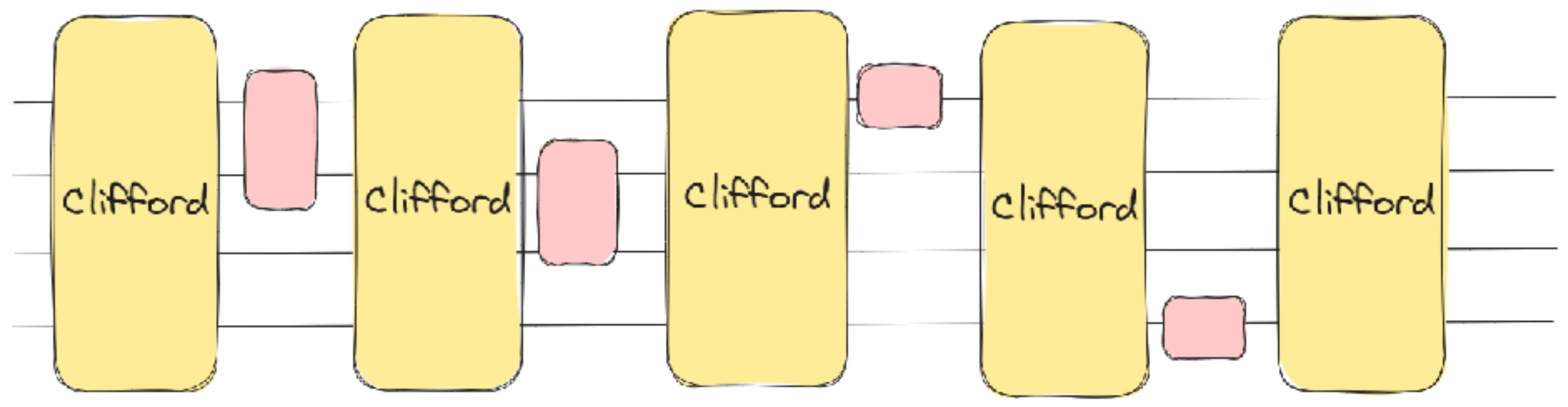
S. Bravyi and D. Gosset, PRL 116 (250501)

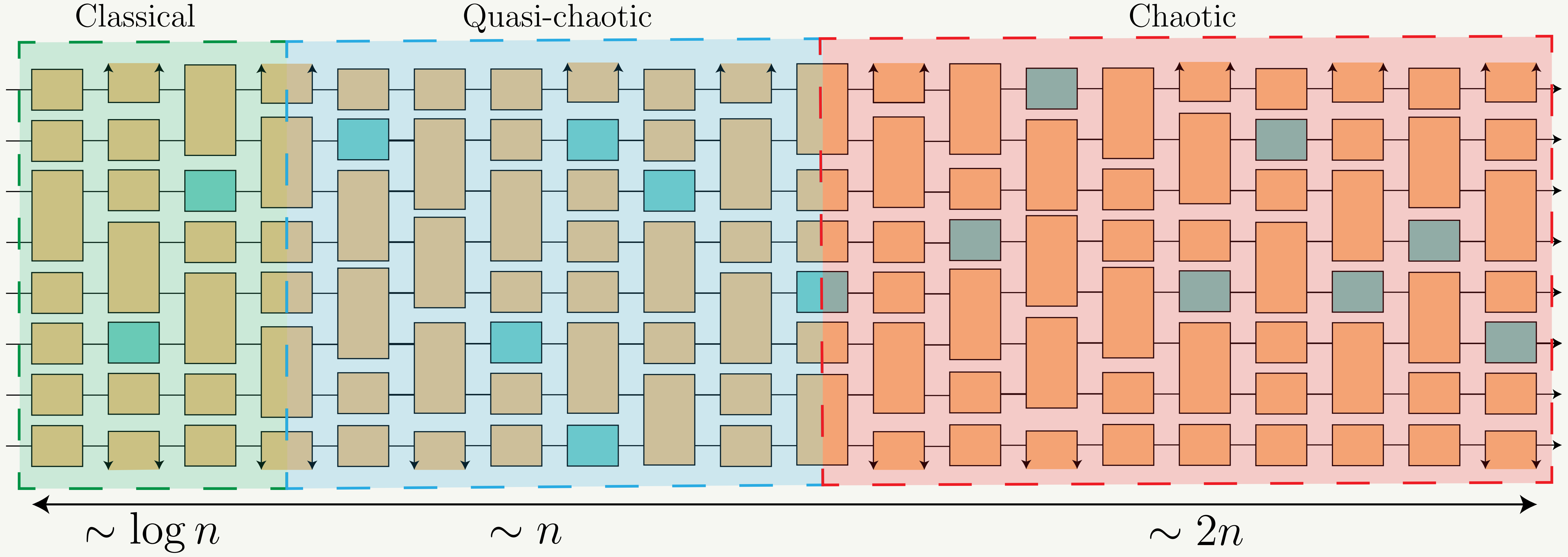
Clifford operations, computational basis states: STABILIZER resources

The t-doped random Clifford circuit

transition to quantum chaos

- A T-doped random Clifford circuit is a circuit made with the benign gates H,P,C_X injected with some T-gates
- As the number *t* of T-gates increases, we inject more **SE** in the circuit
- Its quantum complexity increases, driving a transition from Clifford circuits to more generic circuits





Quantum Chaos is Quantum
L. Leone, S.F.E. Oliviero, Y. Zhou, AH
Quantum 5, 453 (2021)

Magic (non-Stabilizerness)

What is magic?

Magic quantifies the distance from the Free (Clifford) resources

$$M_{\mathrm{dist}}(\psi) := \min_{\sigma \in \mathcal{F}} \frac{1}{2} \|\psi - \sigma\|_1$$

Unfortunately such distances are uncomputable and unmeasurable

Free States for stabilizer formalism

$$\mathrm{PSTAB} = \{C|i_1 \dots i_n\rangle \ : C \in \mathcal{C}(d^n)\}$$

$$\mathrm{STAB0} := \{\text{states purified in PSTAB}\}$$

$$\mathrm{STAB1} := \{\mathrm{Hull}[\ \mathrm{PSTAB}]\}$$

One can establish STAB0 as

$$\mathrm{STAB0} := \{\psi : m_2(\psi) = 0\}$$

m2 is an entropic measure of magic

Non-stabilizerness = String entropy

- Strings proliferate because of non Clifford unitaries
- The linear combination of strings comes with a weight. This weight represents the probability that such a string would be the outcome of a measurement

Example: consider the evolution of a string with two T operations

$$X_1 I_2 \underbrace{T_3 X_3 T_3^\dagger}_{\frac{1}{\sqrt{2}}(X-Y)} Z_4 Y_5 \underbrace{T_k X_k T_k^\dagger}_{\frac{1}{\sqrt{2}}(X-Y)} \dots Y_n$$

String entropy = Stabilizer Entropy SE

$$S_\alpha(\mathbf{p}) = \frac{1}{1-\alpha} \log \sum_x p^\alpha(x)$$

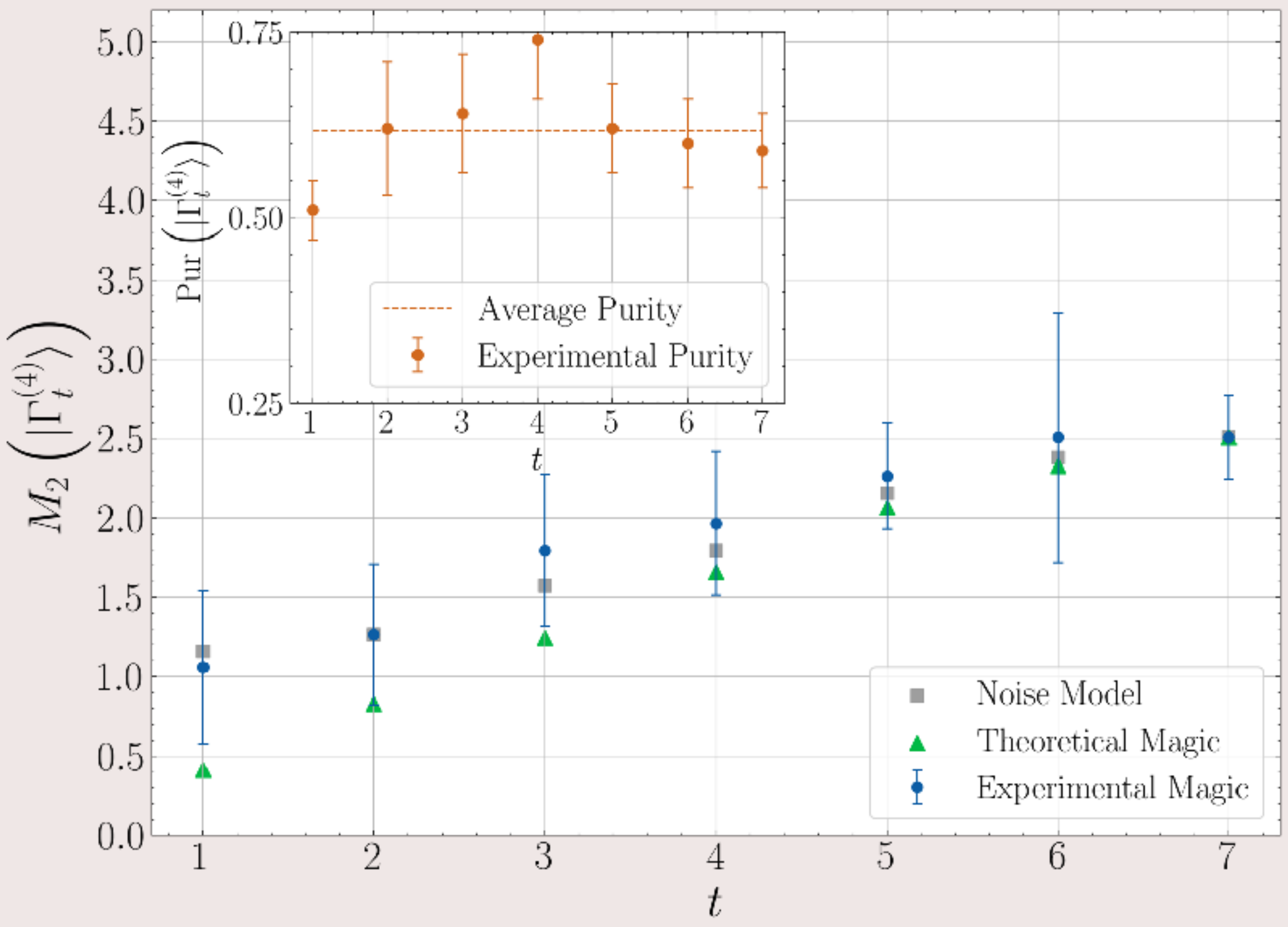
$$\rho = \frac{1}{d} \sum_P \mathrm{tr}(P\rho)P$$

$$m_2(\rho) = -\log\left(\frac{d^{-1}\sum_P \mathrm{tr}^4(P\rho)}{\mathrm{Pur}\rho}\right)$$

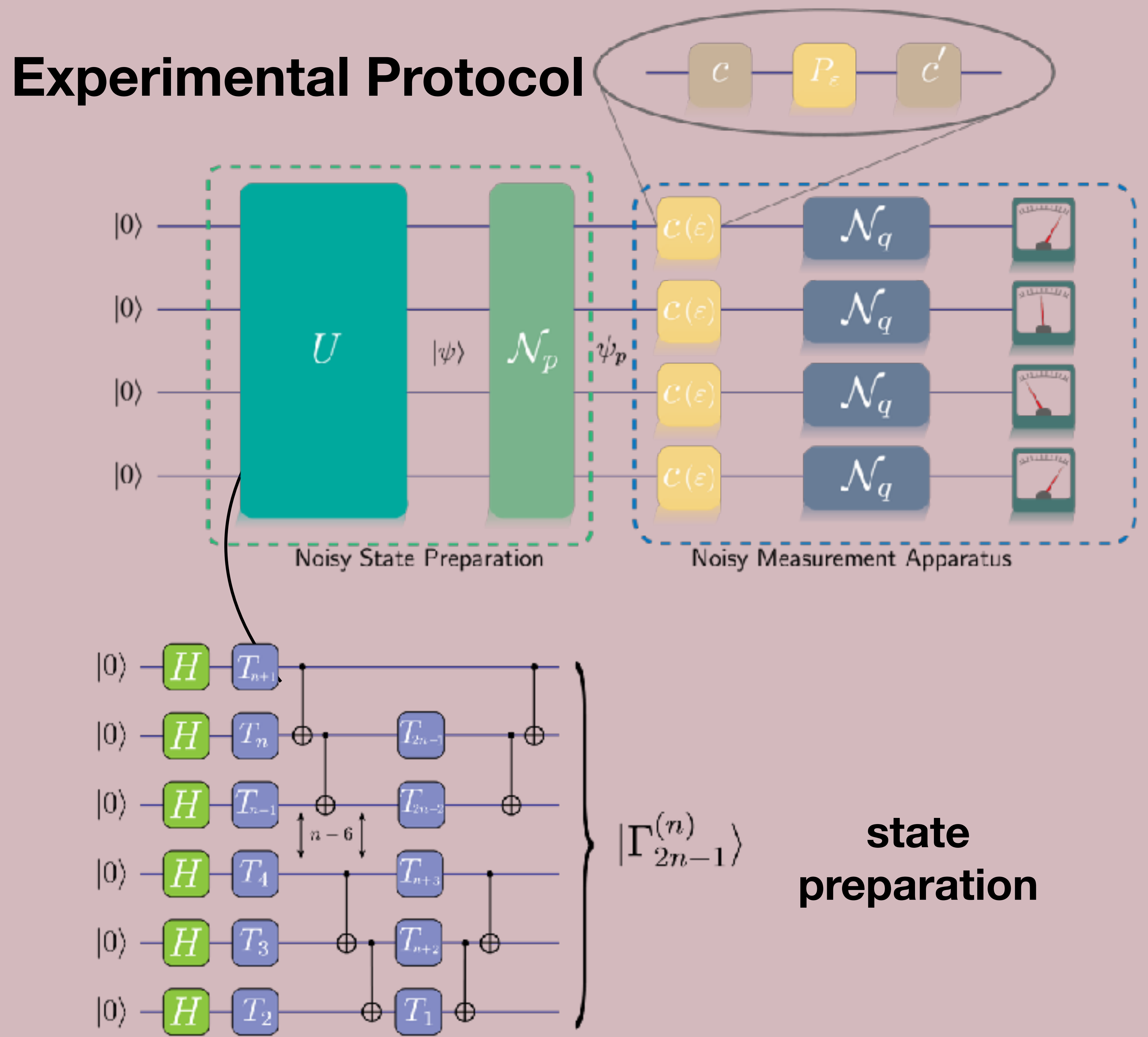
SE is a good monotone for pure states and a good *proxy in the resource theory for mixed states*

Stabilizer Rényi Entropy
L. Leone, S.F.E. Oliviero, AH
Physical Review Letters, 128, 050402 (2022)

SE can be measured



Experimental results IBM-Falcon quantum computer

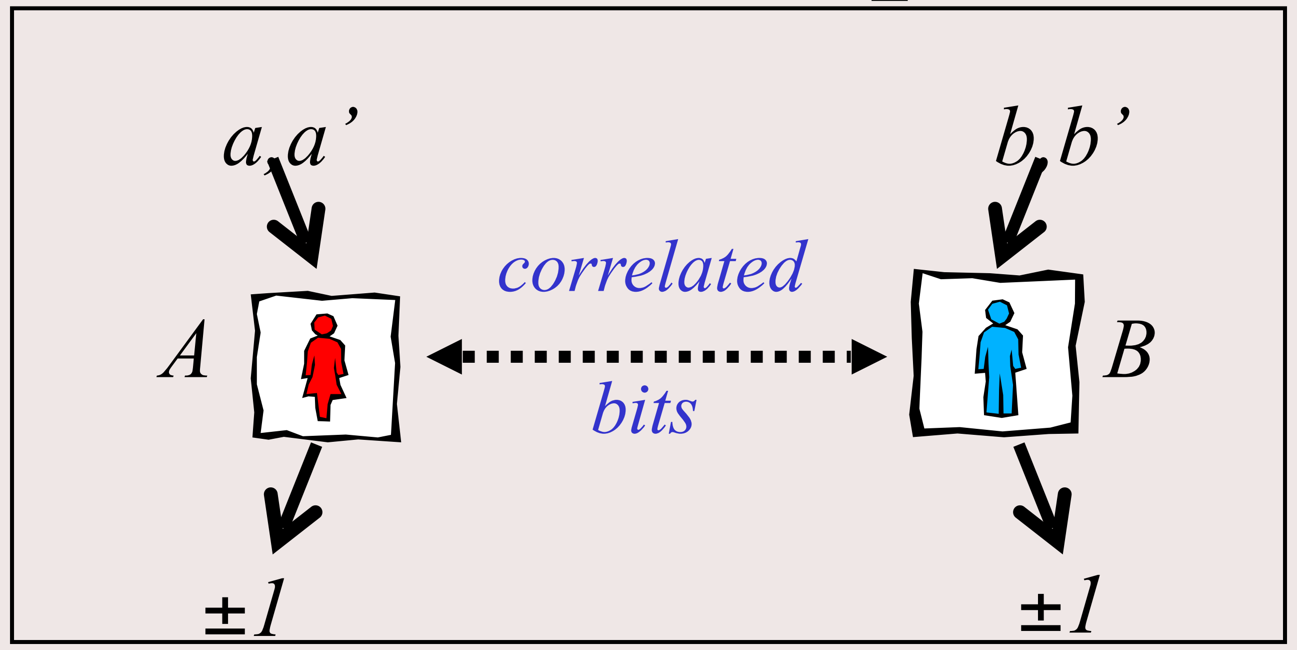


Measuring Magic on a Quantum Processor
L. Leone, S.F.E. Oliviero, AH, S. Lloyd
Nature Physics Quantum Inf 8, 148 (2022)

Interplay between SE and Entanglement

Bell's inequalities, anti-flatness, scrambling

Bell's inequalities



$$|\langle ab\rangle + \langle a'b\rangle + \langle ab'\rangle - \langle a'b'\rangle| \leq 2$$

Let B_0 be made of just Pauli measurements for both Bob and Alice

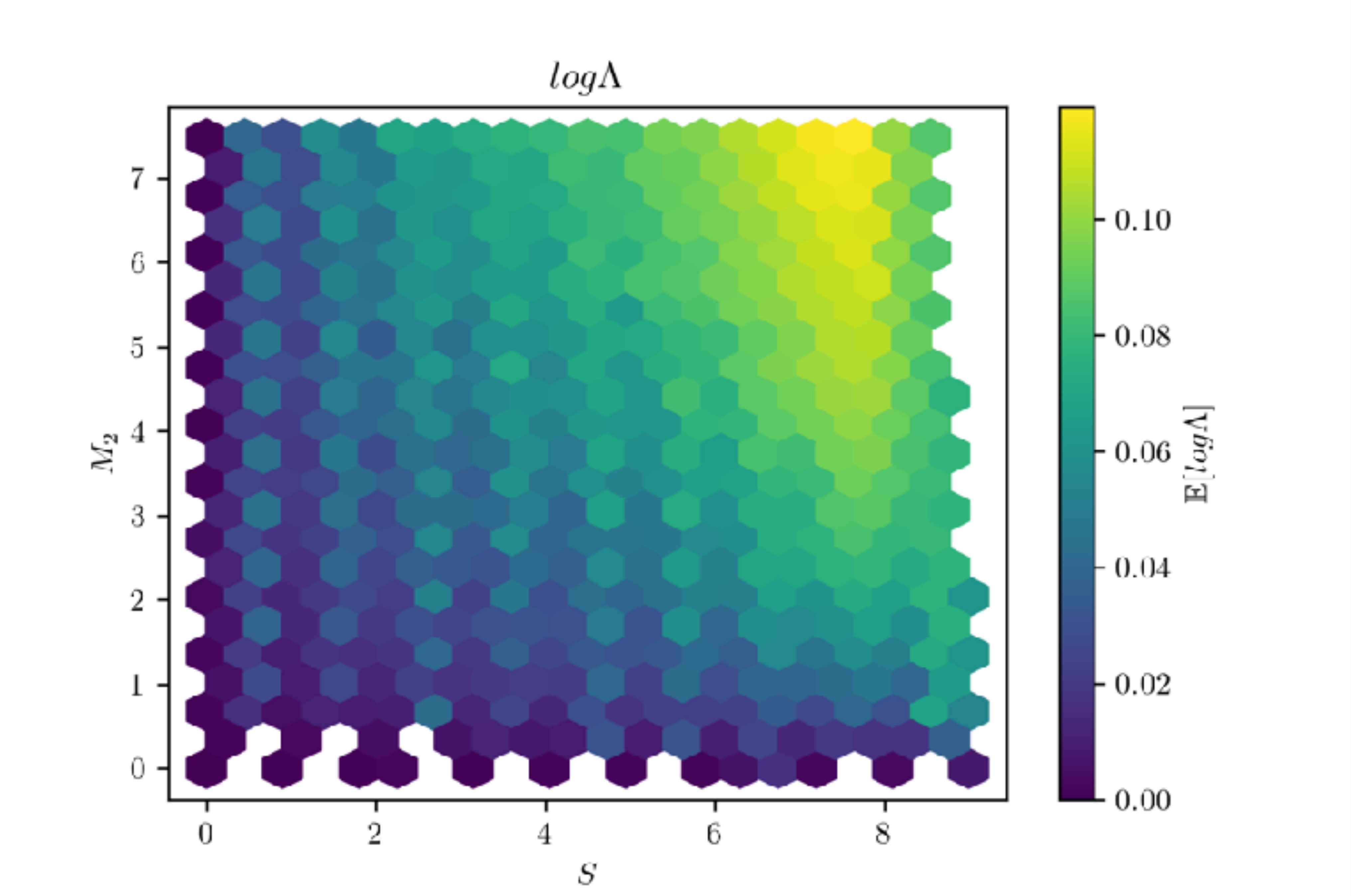
$$\sup_{C} tr\left(B_0 C\omega_0 C^{\dagger}\right) \leq 2$$

$$\Delta B = cm(U)$$

This state can be maximally entangled!

Bell's violation is measured by SE

anti-flatness = scrambled magic



the interplay between SE and entanglement gives rise to scrambled magic and quantum complexity

$$\Lambda_A = \frac{Tr[\rho_A^3]}{(Tr[\rho_A^2])^2}$$

We propose scrambled, non-local magic as a probe and measure of the resource of quantum complexity in many-body systems

$$m^{NL}(\psi) = \min_{U_A \otimes U_B} m_2\left[U_A \otimes U_B(\psi)\right]$$

Scrambling

1. Quantum chaos depends on the *quantum butterfly effect*: far away influences can have dramatic effects everywhere

2. In order for a perturbation to spread everywhere, its operator needs to fragment and grow: it is a combination of entanglement and string entropy, that is, ***SE***

3. If there is chaos, there is scrambling, and scrambling can be measured by the decay of correlations: OTOC

OTOC:

$$\Omega_{AD}(U_t) = \frac{1}{2^n}\langle \mathrm{tr}(P_A U_t^\dagger P_D U_t P_A U_t^\dagger P_D U_t)\rangle$$

scrambling condition:

$$\Omega_{AD}(U_t) \simeq \frac{1}{2^{2|A|}} + \frac{1}{2^{2|D|}} - \frac{1}{2^{2(|A|+|D|)}}$$

$$M_\alpha(U) = \frac{1}{1-\alpha}\log \Omega_\alpha(U)$$

quantum chaos is reached when SE is maximal

Quantum Chaos is Quantum
L. Leone, S.F.E. Oliviero, Y. Zhou, AH
Quantum 5, 453 (2021)

Magic and quantum protocols

verification, learning, decoding black holes

Quantum certification

1. What do we need to achieve quantum advantage? A magic M scaling with the number of qubits

2. Quantum computers are notoriously fragile. One needs to check if they are doing what they promise, measuring a *fidelity*

3. Measuring the fidelity turns to be harder as the quantum computer is more advantageous, another sign of fragility!

quantum complexity = hardness in certification

The number of resources N_ψ needed to ascertain the fidelity $\mathcal{F}$ with accuracy ε and success probability $1-\delta$ is bounded as

$$\frac{2}{\epsilon^2}\ln\frac{2}{\delta}\exp[M_2(\psi)] \leq N_{\tilde{\psi}} \leq \frac{64}{\epsilon^4}\ln\frac{2}{\delta}\exp[M_0(\psi)]$$

Nonstabilizerness determining the hardness of direct fidelity estimation
L. Leone, S.F.E. Oliviero, AH
Phys. Rev. A 107, 022429

Learning t-doped stab states

L Leone, SFE Oliviero, AH
Learning t-doped stabilizer states
Quantum 2024-05-27, volume 8, page 1361

- Given a t-doped Clifford Circuit, a t-doped stabilizer state is defined as
$$|\psi_t\rangle \equiv C_t|0\rangle^{\otimes n}$$
- For any t
$$|\psi_t\rangle\langle\psi_t| \equiv 2^{t-n}\underbrace{\left(\mathbb{I}+\sum_{i=1}^{k}\mathrm{tr}(h_i|\psi_t\rangle\langle\psi_t|)h_i\right)}_{H}\underbrace{\left(\prod_{j}^{|G_{\psi_t}|}\left(\frac{\mathbb{I}+\phi_j g_j}{2}\right)\right)}_{\Pi}$$
- Elements of H and Π commute.
- For $t<n$ it holds that: $|\psi_t\rangle \equiv \tilde{D}_{G(C_t)}|\phi_t\rangle \otimes |0\rangle^{n-t}$

learning by compressing

the stabilizer entropy requires

$$\mathcal{O}(\exp(t)\,\mathrm{poly}(n))$$

Quantum correlations

spatial correlations:

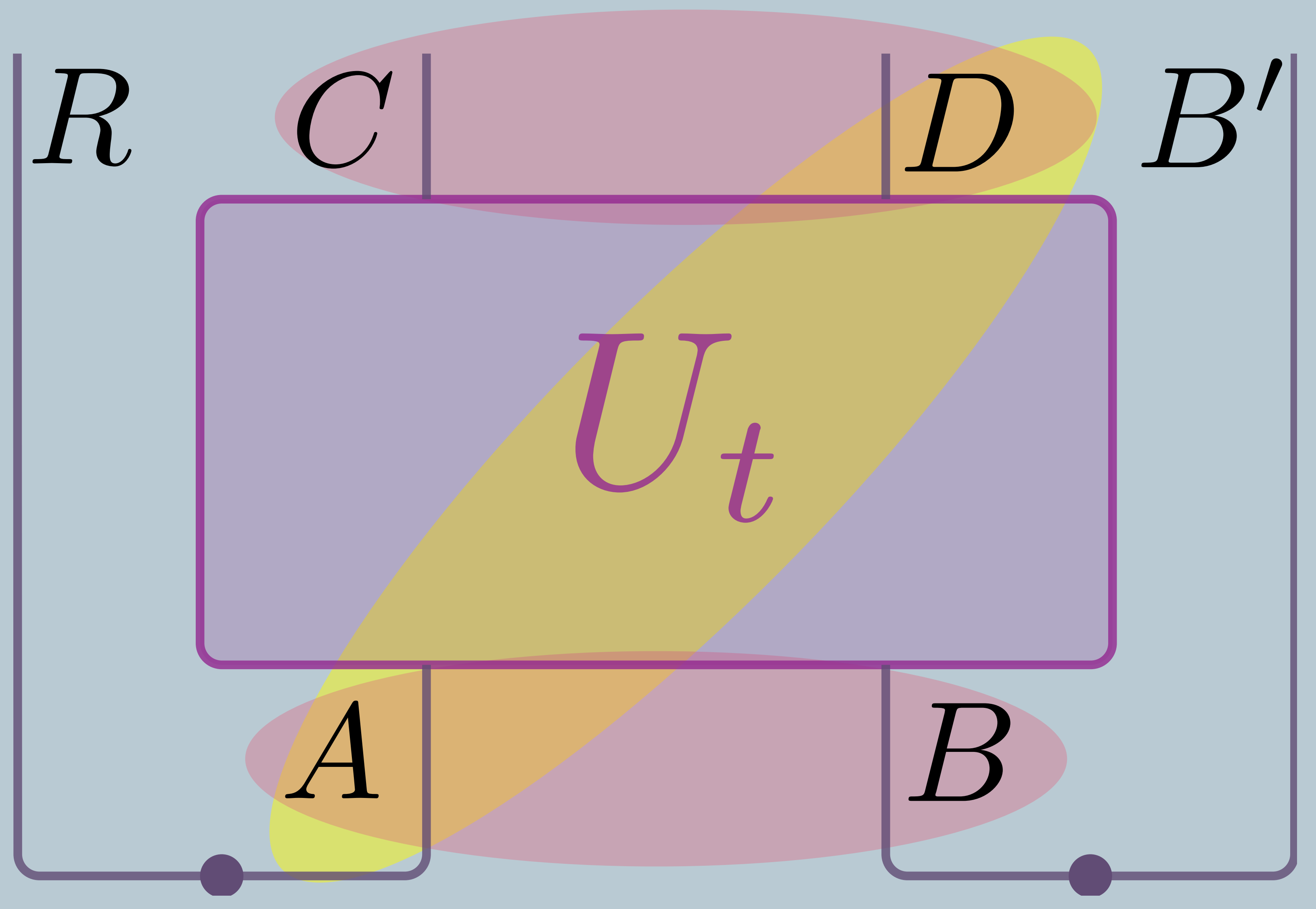
$$I(A|B) = S(A) + S(B) - S(AB)$$

space-time correlations:

$$I(A|D) = S(A) + S(D) - S(AD)$$

All the information of A is encoded by the entanglement with R

$$|\Psi_t\rangle \equiv U_t|BB'\rangle|AR\rangle$$



The channel *E* goes from A to DB'

$$\mathcal{E} : |A\rangle \mapsto \mathrm{tr}_{RC}\Psi_t$$

Decoupling theorem

$$I(R|DB') \equiv |A| + \log 2^{2|A|}\Omega_{AD}(U_t)$$

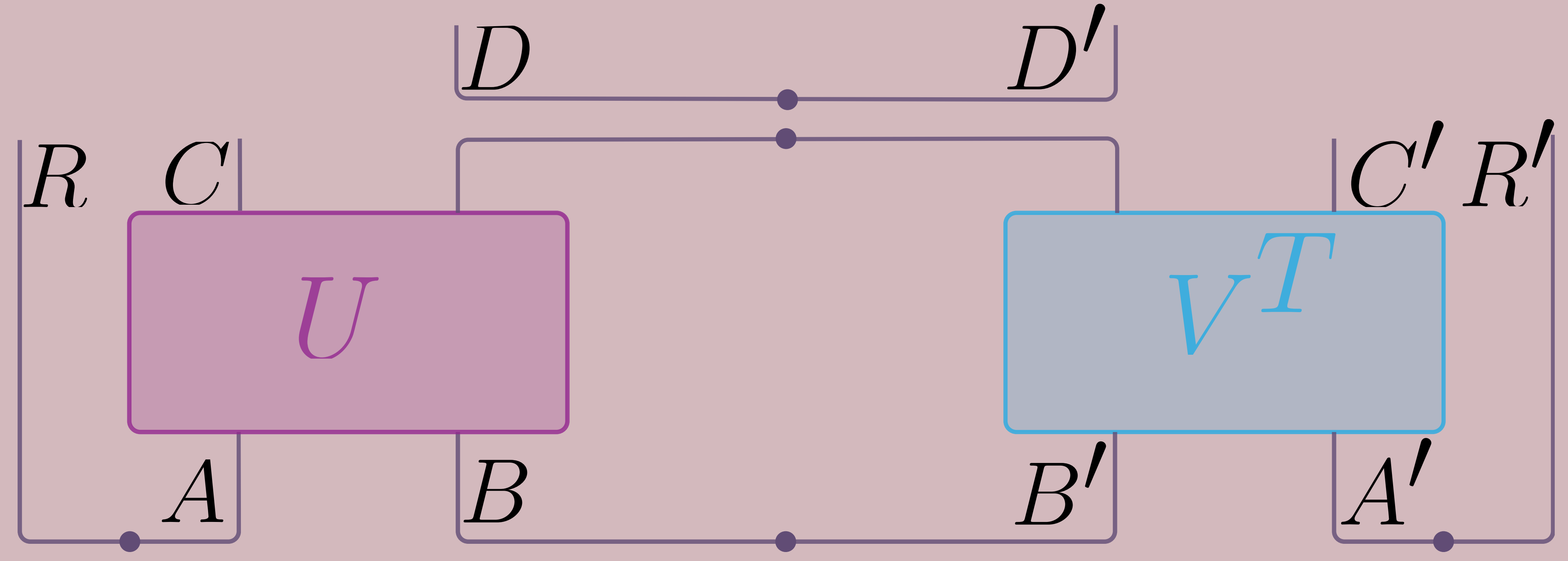
So if the U is scrambling A is perfectly correlated with the output DB':

$$I(R|DB') = |A| - \epsilon$$

$$|D| = |A| + \log \epsilon^{-1/2}$$

How do we recover |RA'> from

the output DB'?



We need U = V^T for this to work, but U is actually inaccessible. What can we do? can we learn U?

Hayden-Preskill, JHEP 2007, 120

Kitaev-Yoshida, 1710.03363

Efficient decoder learning

Imagine there are some preserved Clifford orbits:

$$G_\Lambda(U_t) := \{P \in \mathbb{P}_\Lambda \,|\, U_t^\dagger P U_t \in \mathbb{P}_n\}$$

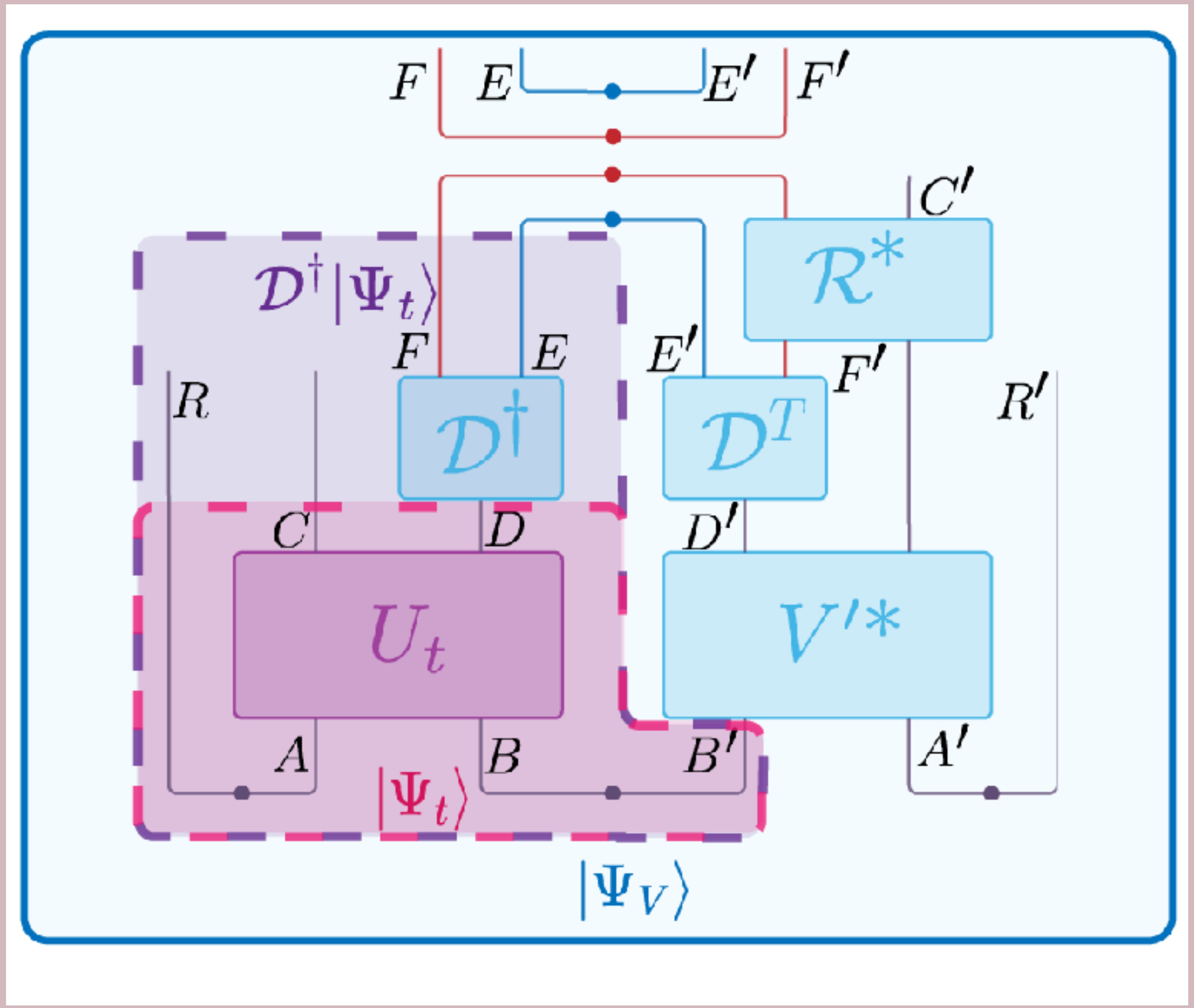
$$P \in G_D(U_t), \quad V^\dagger P V = U_t^\dagger P U_t$$

The randomizer R completes V in a random way, keeping it Clifford

$$\mathcal{F}_V(U_t) \simeq \frac{1}{1+\frac{d_A^2-1}{|G_D(U_t)|}} \geq \frac{1}{1+2^{2|A|+t-2|D|}}$$

perfect fidelity for t<n; then degradation occurs

$$\mathcal{P}(V) \geq 1 - 2^{t-2(n-|D|)}$$



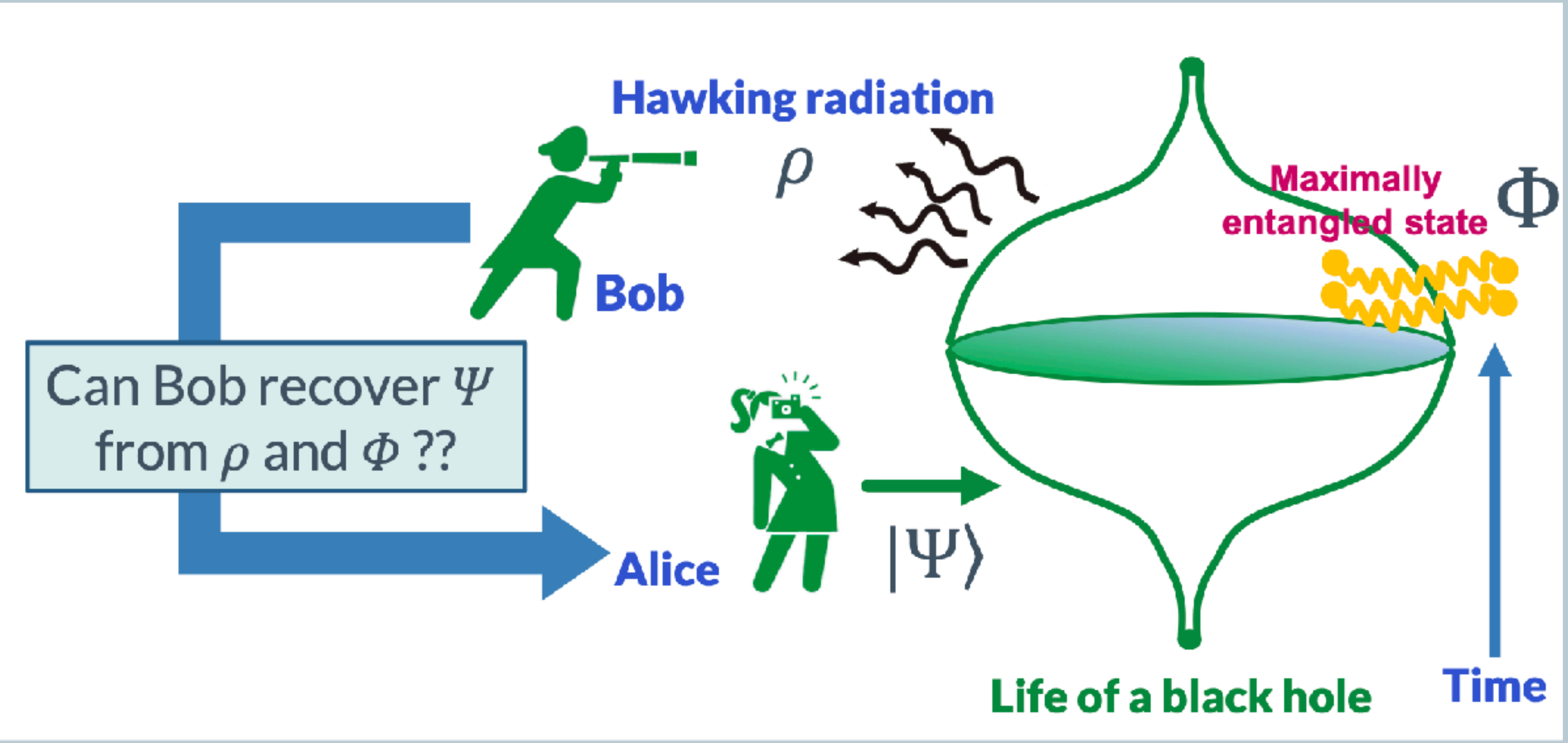
We effectively pushed SE in the subsystem F of D

It is *very important* to notice that V is not U!

L Leone, SFE Oliviero, S Lloyd AH
Phys. Rev. Lett. 132, 080402 (2024) and
Phys. Rev. A 109, 022429 (2024)

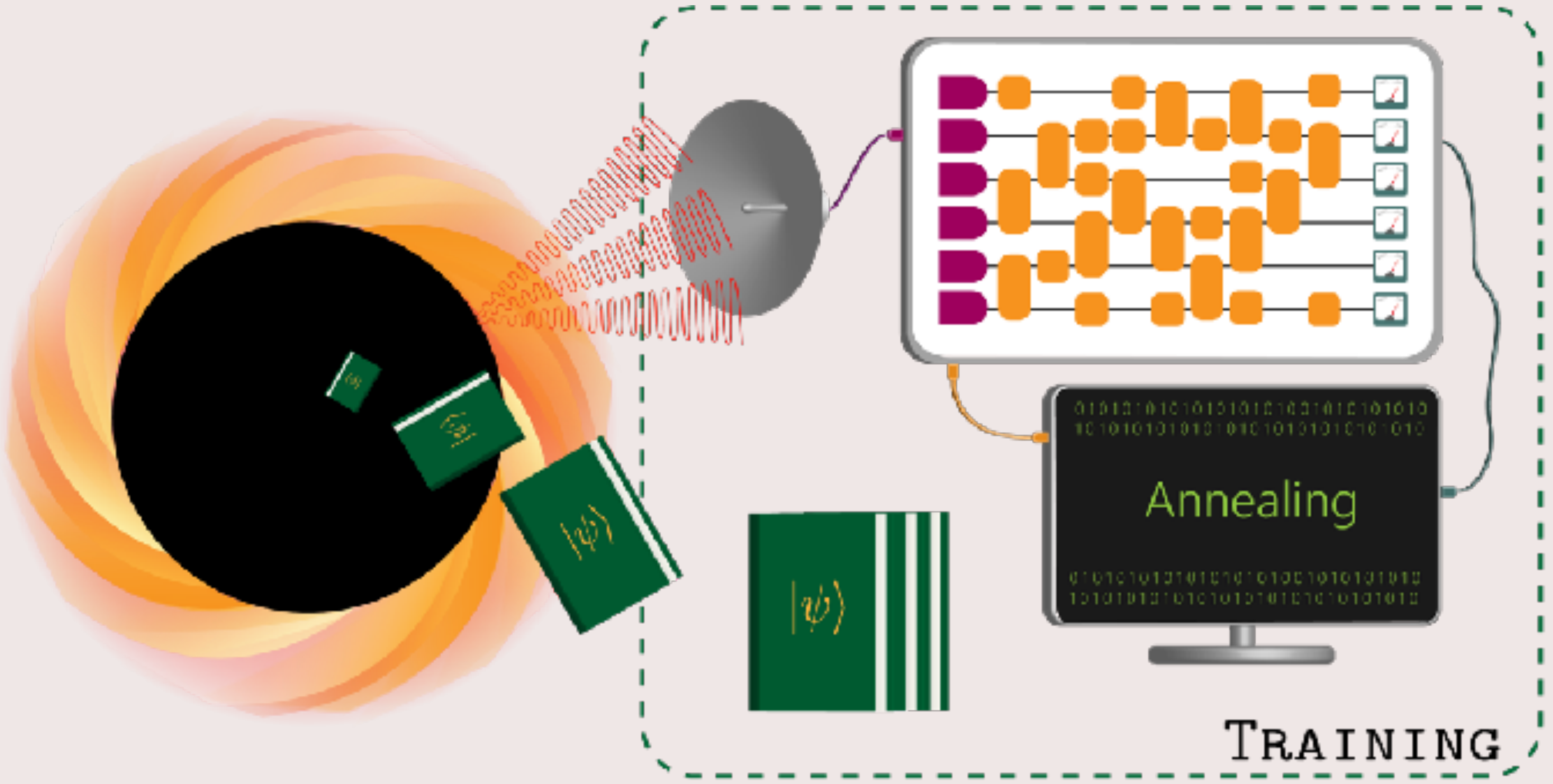
Black holes are scramblers that leak information

The black hole is an information scrambler. Recovering the information requires to be able to unscramble



“Alice laughed. 'There's no use trying,' she said. 'One can't believe impossible things.’

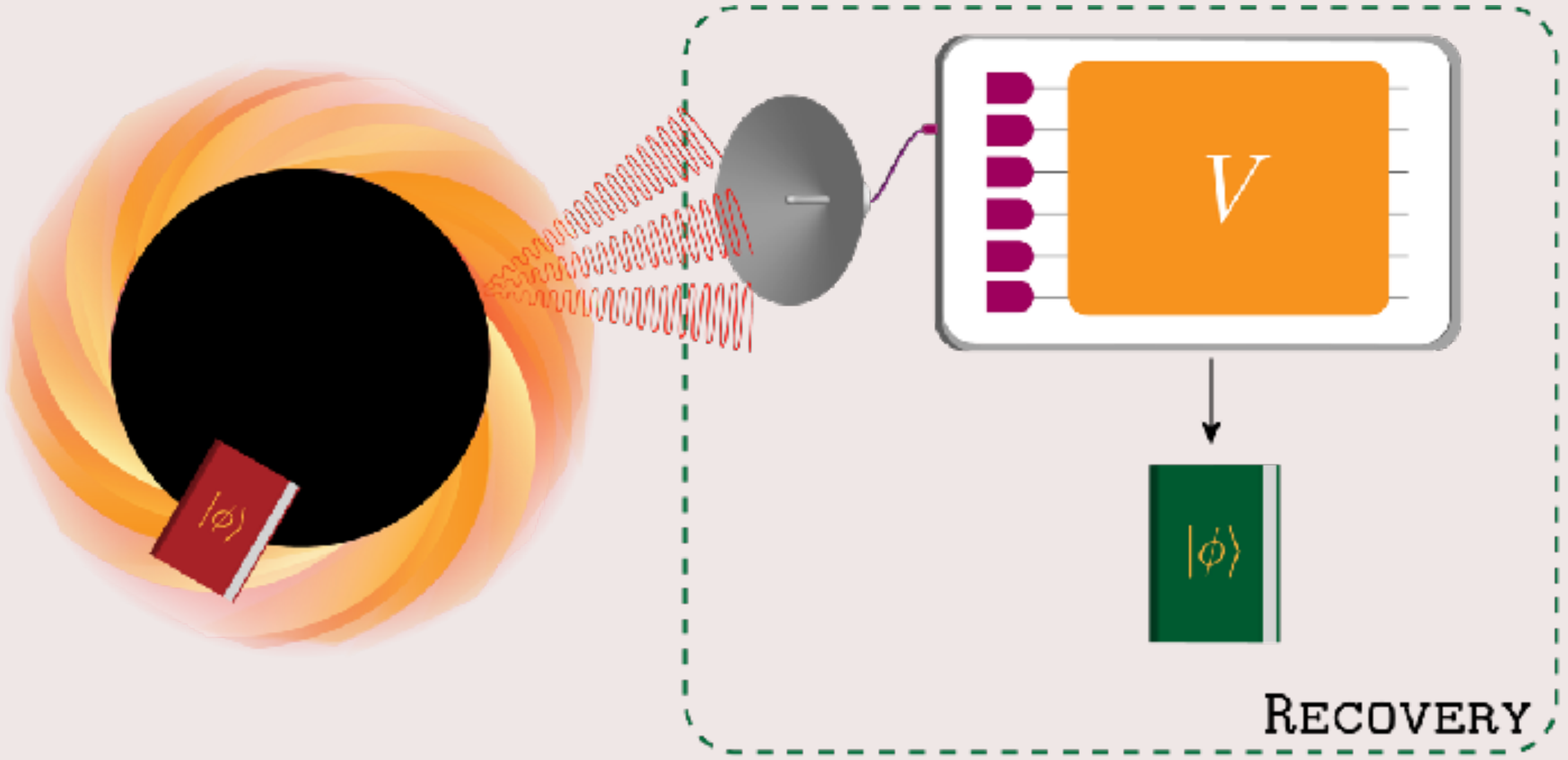
I daresay you haven't had much practice,' said the Queen. 'When I was your age, I always did it for half-an-hour a day. Why, sometimes I've believed as many as six impossible things before breakfast.



The price for learning is

$$\mathrm{poly}(n, 2^t)$$

as long as t<n



However, the price for simulating the black hole is poly(n)! How can a complex U be decoded by a simple U? This sounds quite incredible!

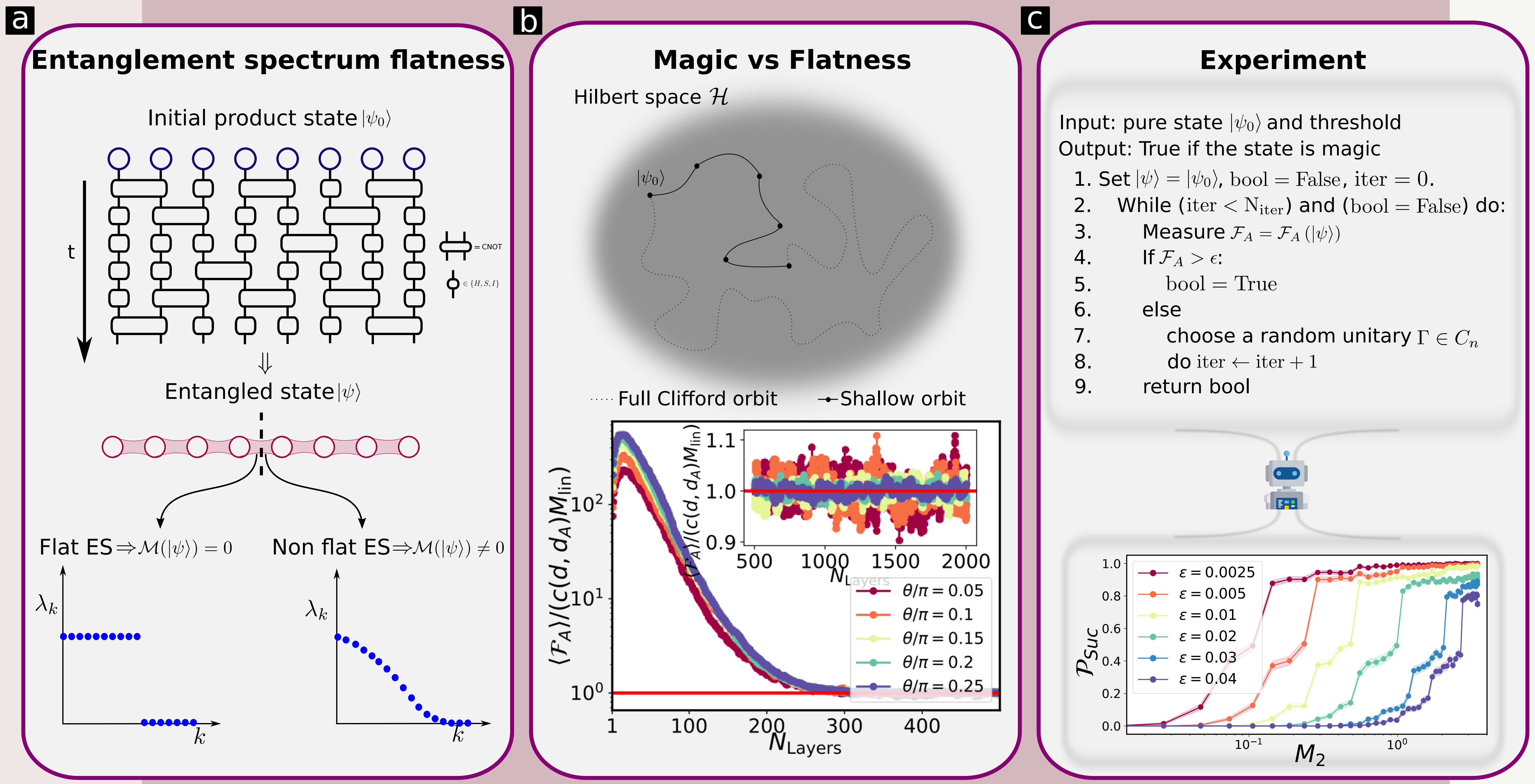
SE and EE in quantum many-body systems

SE and entanglement

Anti-flatness of the entanglement spectrum: entanglement is needed to have a spectrum to start with

On the other hand, *magic*, makes the spectrum bend: non magic = FLAT

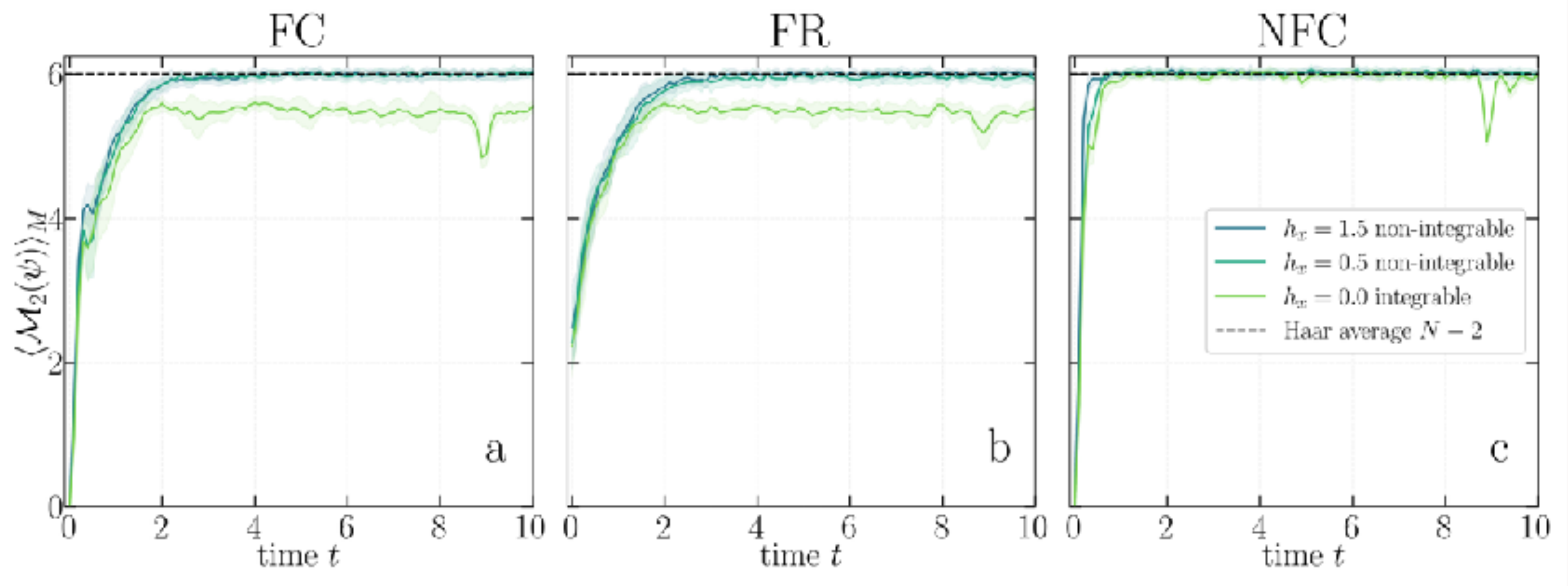
Average on C is actually magic!

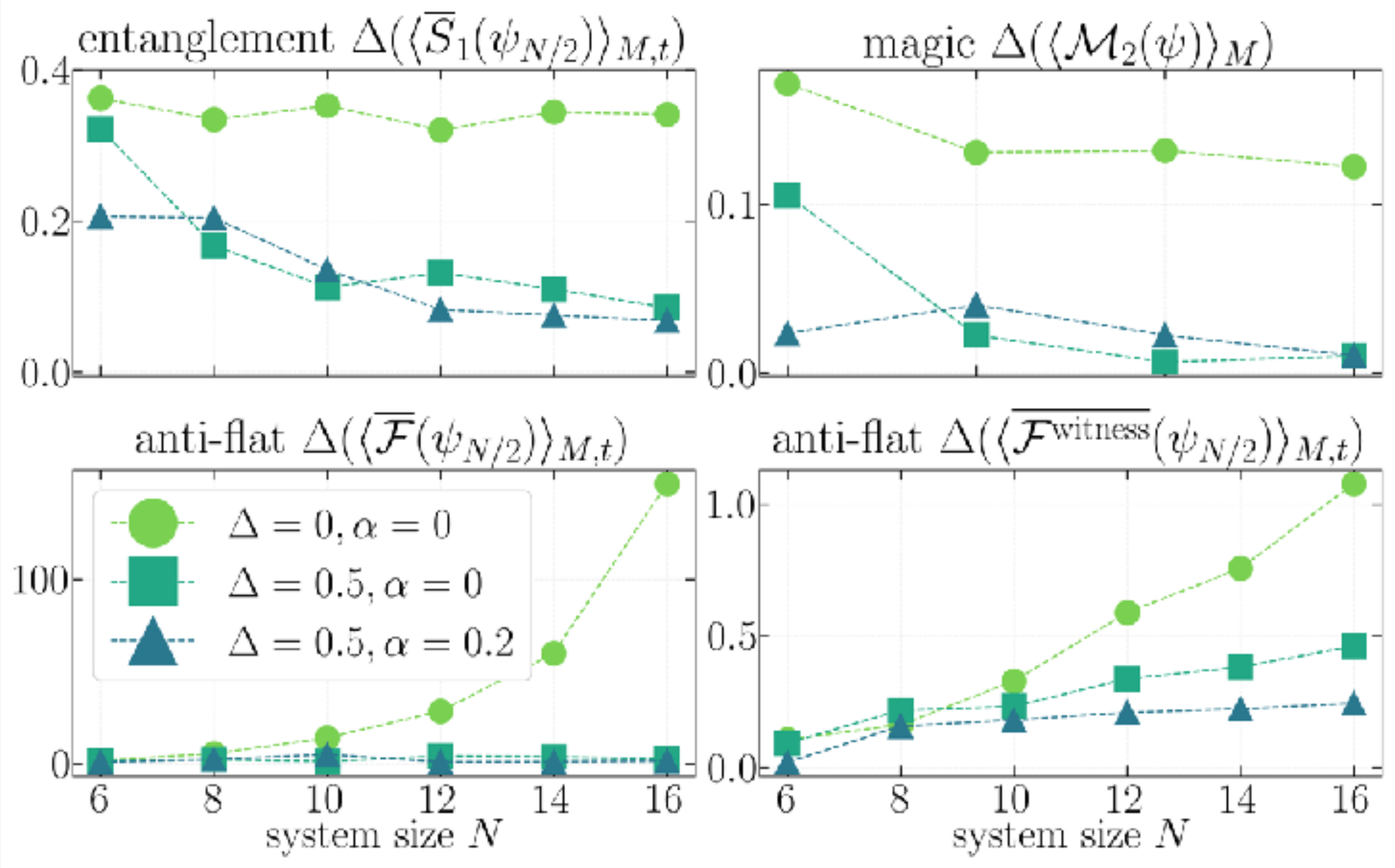


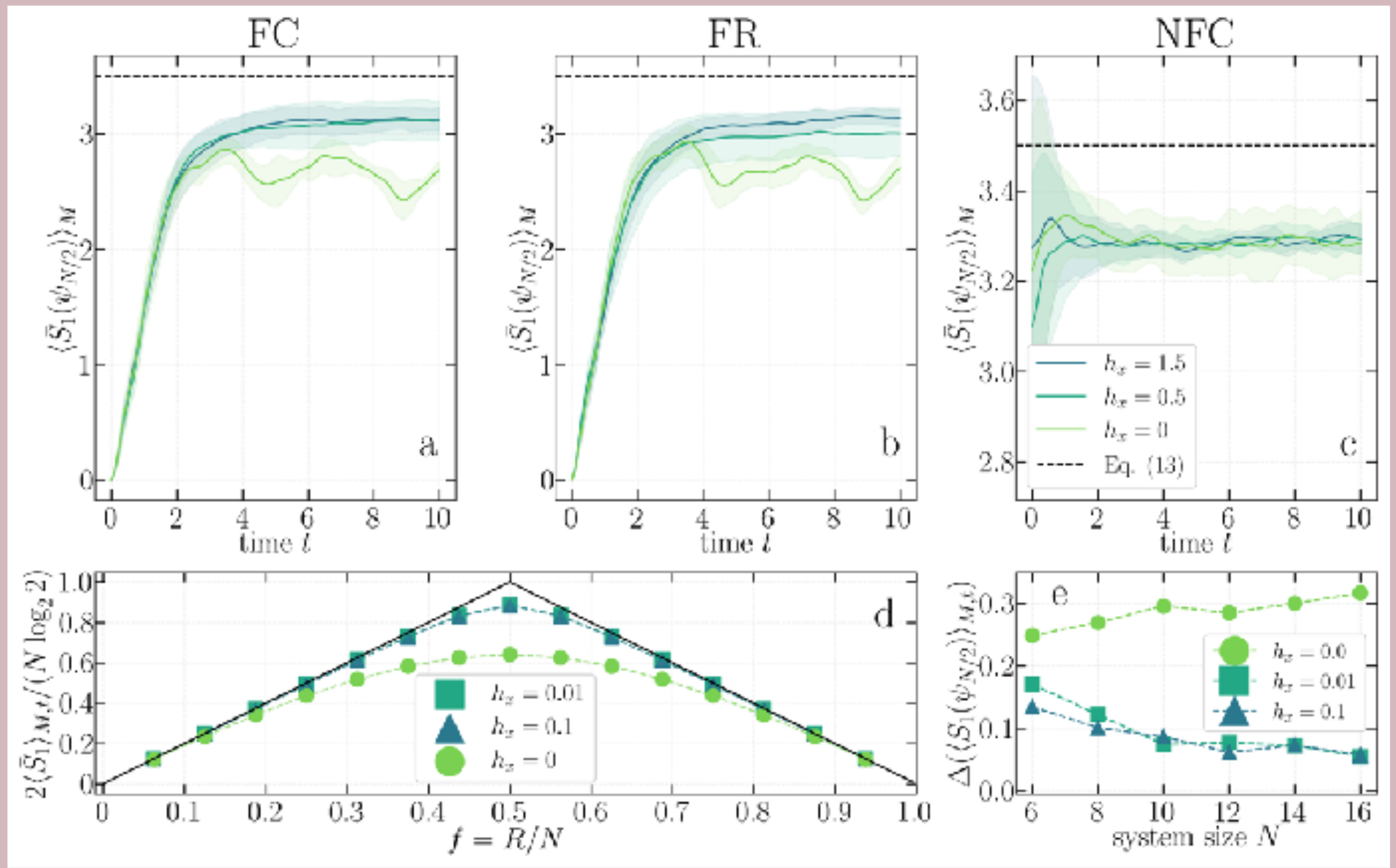
Phys. Rev. A 109, L040401 (2024)

$$\mathcal{F}_A(\psi) = \mathrm{Tr}\psi_A^3 - \mathrm{Tr}^2\psi_A^2$$
$$\langle \mathcal{F}_A(C\psi)\rangle_C = \alpha(d, d_A) M_{lin}(|\psi\rangle)$$

SE in quantum spin chains







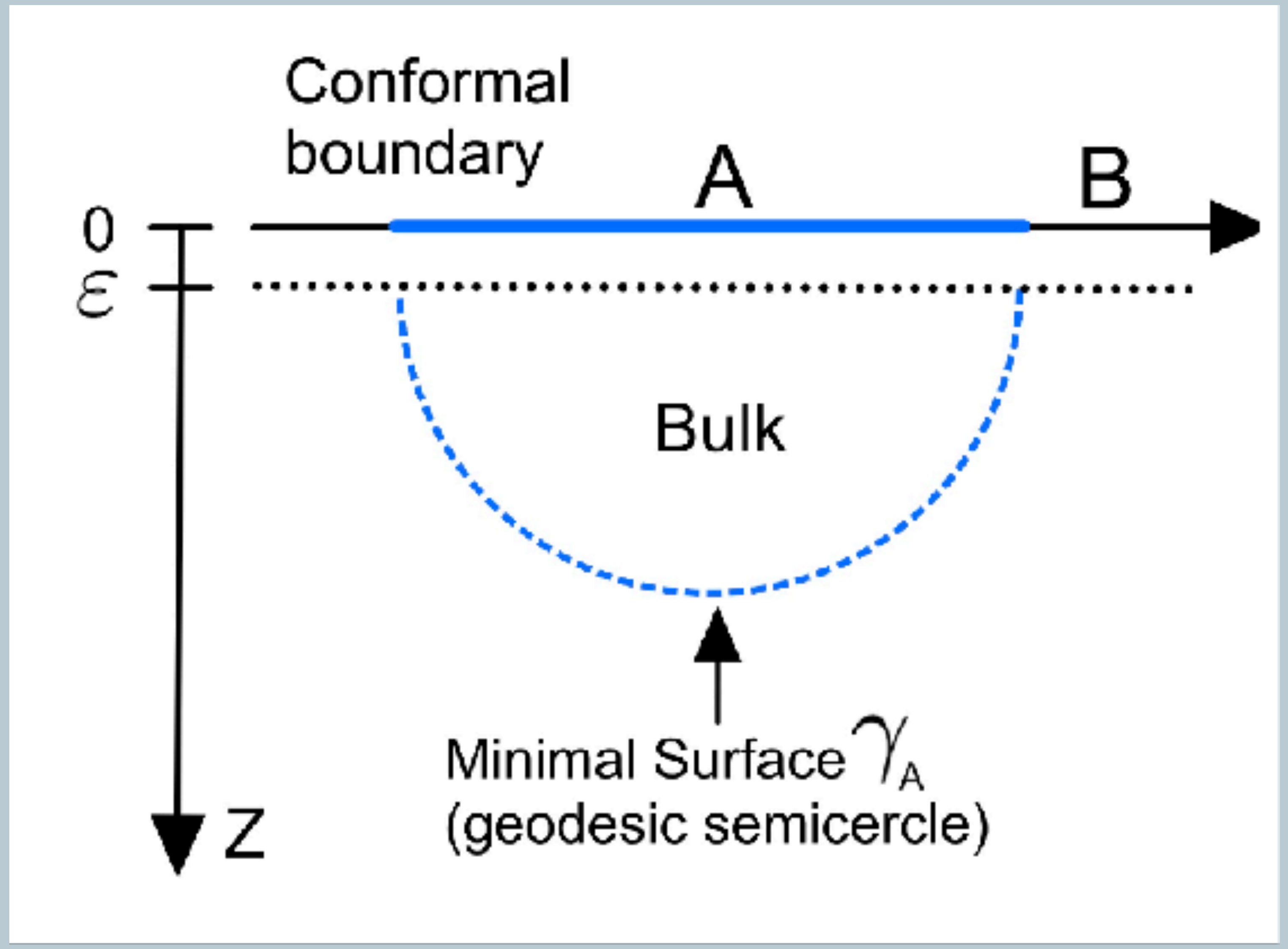
A stabilizer gap separates integrable from non integrable systems

work in progress...

Holography and scrambled magic

AdS-CFT correspondence

- Holography states that quantum field theory on the boundary is equivalent to GR in the bulk
- Breakthrough: RT formula. Entanglement in CFT (quantum) is equivalent to minimal surface in AdS (solution to Einstein equations)
- But what about gravity? Gravity needs back-reaction. This can be measured to the susceptibility of areas to matter = *tension*



Big Question:

what is the holographic dual of back-reaction?

The holographic dual of gravity is magic

- The intuition is: by RT formula, we know that surfaces are Rényi entropies, and tension are susceptibility to the Rényi index n
- figure of speech: without tension, a drum gives no sound.
- However, without SE, all spectra must be flat: somehow SE is involved in tension and thus in back-reaction

C Cao, G Cheng, AH, L Leone, W Munizzi, SFE Oliviero, 2403.07056

local unitary transformations $U_A \otimes U_B$ cannot change the spectra so have no holographic meaning. What then?

$$\frac{1}{(4G)^2}\frac{\partial \gamma_A}{\partial \mathcal{T}} \propto m^{NL}(\psi)$$

the holographic dual of back-reaction is (*non-local) magic*

(partial) list of open problems

- Scrambling in shallow t-doped circuits
- SE as purification resource
- correlations between SE and EE
- resource theory of non-local magic
- non-local magic susceptibility and topological order
- SE in the SYK model
- stabilization and disentangling in temperature
- gravity and non local magic in AdS-CFT
- SE and non-locality/non-realism